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A General Theory of Aircraft Response to Three-Dimensional Turbulence

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An improved mathematical description of the dynamic response of an aircraft to three-dimensional isotropic turbulence is developed by establishing a unique correspondence between the spatial symmetry properties of the ambient field of turbulence and those of the aircraft, thereby yielding a more elegant and compact formulation offering significant advantages over the direct multiple input approach: 1) a 36-fold reduction in the size of the quadratic form that characterizes the response power spectral densities; 2) complete analytical separation between symmetric and antisymmetric response; 3) insight into the nature and origin of responses arising from interaction between longitudinal, lateral, and vertical gust velocity input components; 4) greater compatibility with conventional dynamic response computational procedures. The formulation is extended to include transfer functions, cross transfer functions, and coherence functions, such as are typically derived from dynamic response flight tests in which a gust reduction system is employed.

I. Introduction

THE accelerated development of large, subsonic aircraft and the availability of digital computers of increasing speed and capacity, has stimulated renewed interest in the problem of representing the full three-dimensional spatial dependence of the gust environment in dynamic response calculations. A straightforward multiple input treatment of the aircraft, as exemplified by Lin[1] and Fuller,[2] restricts physical insight and incurs a 36-fold redundancy in the mathematical description of the gust environment, which accordingly reduces computational efficiency. The formulation presented here avoids this redundancy and yields a comprehensive theory of gust response by employing two-point gust velocity input configurations to resolve the turbulence velocity field into two independent subfields, one symmetric and the other antisymmetric with respect to the aircraft geometry. Each subfield, by inducing structural responses of corresponding symmetry only, serves to maintain the conventional analytical separation between symmetric and antisymmetric response. Relevant properties of the turbulence field are derived by systematically introducing the assumptions of homogeneity, stationarity, isotropy, and Taylor's hypothesis. The following development thus may provide a point of departure for the investigation of cases involving modification of these assumptions.

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II. Decomposition of the Turbulence Field

Let $\mathbf{u}(\mathbf{r},t)$ represent the gust velocity at point $\mathbf{r}$ and time t, where $\mathbf{r}$ has coordinates (x,y,z) relative to an inertial frame. Reflection of a velocity vector about its local u_1,u_3 plane, and reflection of a position or separation vector through the x,z plane of the inertial frame will be indicated by affixing a prime. Thus, $\mathbf{u}' = (u_1,-u_2,u_3)$, and $\mathbf{r}'$ represents the point $(x,-y,z)$, as shown in Fig. 1. The relations between the components of the primed and unprimed vectors may be conveniently expressed as follows:

$$v'_i = (-1)^{i-1}v_i \tag{1}$$

where $\mathbf{v}$ is any vector.

Like Cartesian components of the gust velocity, impinging upon aerodynamic surface panels at bilaterally symmetric locations on a moving aircraft, eventually will be related to symmetric and antisymmetric two-point velocity configurations, which are denoted by $\mathbf{u}^+(\mathbf{r},t)$ and $\mathbf{u}^-(\mathbf{r},t)$, respectively, and are defined by

$$\mathbf{u}^{\pm}(\mathbf{r},t) = [\mathbf{u}(\mathbf{r},t) \pm \mathbf{u}'(\mathbf{r}',t)]/2 \tag{2}$$

so that

$$\begin{aligned} \mathbf{u}(\mathbf{r},t) &= \mathbf{u}^+(\mathbf{r},t) + \mathbf{u}^-(\mathbf{r},t) \\ \mathbf{u}'(\mathbf{r}',t) &= \mathbf{u}^+(\mathbf{r},t) - \mathbf{u}^-(\mathbf{r},t) \end{aligned} \tag{3}$$

Consider the ensemble averages of all possible products between velocity components $u^{\pm}_i(\mathbf{r}_m,t)$ and $u^{\pm}_j(\mathbf{r}_n,t+\tau)$ measured at points $\mathbf{r}_m$ and $\mathbf{r}_n$, and at times t and $t+\tau$, respectively. If the velocity field is homogeneous and stationary, then the ergodic hypothesis may be invoked to replace ensem-

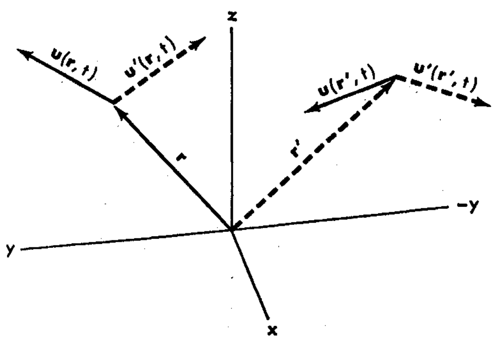

Fig. 1 Vectors used in constructing two-point symmetric and antisymmetric velocity configurations.

ble averages by suitable space averages.[3] Denoting the space averages by means of the brackets $\langle\ \rangle_{\mathbf{q}}$ and applying Eqs. (1) and (2), we then obtain for the product of velocity components of like symmetry:

$$\langle u^{\pm}{}_i(\mathbf{q}+\mathbf{r}_m,t)u^{\pm}{}_j(\mathbf{q}+\mathbf{r}_n,t+\tau)\rangle_{\mathbf{q}} = \tfrac{1}{4}\{\langle u_i(\mathbf{q}+\mathbf{r}_m,t)u_j(\mathbf{q}+\mathbf{r}_n,t+\tau)\rangle_{\mathbf{q}} \pm (-1)^{i-1}\langle u_i(\mathbf{q}+\mathbf{r}'_m,t)u_j(\mathbf{q}+\mathbf{r}_n,t+\tau)\rangle_{\mathbf{q}} \pm (-1)^{j-1}\langle u_i(\mathbf{q}+\mathbf{r}_m,t)u_j(\mathbf{q}+\mathbf{r}'_n,t+\tau)\rangle_{\mathbf{q}} + (-1)^{i+j-2}\langle u_i(\mathbf{q}+\mathbf{r}'_m,t)u_j(\mathbf{q}+\mathbf{r}'_n,t+\tau)\rangle_{\mathbf{q}}\} \quad (4)$$

The product of velocity components of unlike symmetry; i.e., $\langle u^{\pm}{}_i(\mathbf{q}+\mathbf{r}_m,t)u^{\mp}{}_j(\mathbf{q}+\mathbf{r}_n,t+\tau)\rangle_{\mathbf{q}}$ is identical to Eq. (4) except for sign reversal of the last two space averages.

Since the space averages in Eq. (4) depend only upon the difference between the position vectors and explicit time coordinates of the product gust velocity components, they therefore correspond to space-time correlation functions of the form

$$R_{ij}(\mathbf{r}_{mn},\tau) = \langle u_i(\mathbf{q}+\mathbf{r}_m,t)u_j(\mathbf{q}+\mathbf{r}_n,t+\tau)\rangle_{\mathbf{q}} \quad (5)$$

where $\mathbf{r}_{mn} = \mathbf{r}_n - \mathbf{r}_m = (x_n - x_m, y_n - y_m, z_n - z_m)$ denotes the separation vector between the points $\mathbf{r}_m$ and $\mathbf{r}_n$. The space average may be replaced by an equivalent time average[3] by expressing $\mathbf{q}$ as a function of t and averaging over the resulting trajectory. Accordingly, let $\mathbf{q} = \mathbf{V}t$, where $\mathbf{V}$ is constant, and let $\langle\ \rangle_t$ denote a time average. Then

$$R_{ij}(\mathbf{r}_{mn},\tau) = \langle u_i(\mathbf{V}t+\mathbf{r}_m,t)u_j(\mathbf{V}t+\mathbf{r}_n,t+\tau)\rangle_t \quad (6)$$

The space-time correlation function, being a dyadic product of the velocity vectors, is a tensor of second rank. Under the condition of isotropy, such tensors must have the general form[4]

$$R_{ij}(\mathbf{r},t) = F(r,t)r_ir_j + G(r,t)\delta_{ij} \quad (7)$$

where r_i is the ith Cartesian component of $\mathbf{r}$, and δ_{ij} is the

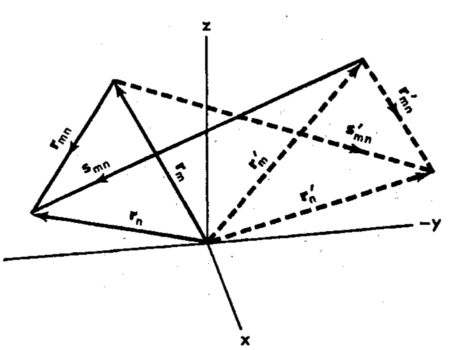

Fig. 2 Separation vectors involved in the correlation between two-point velocity configurations.

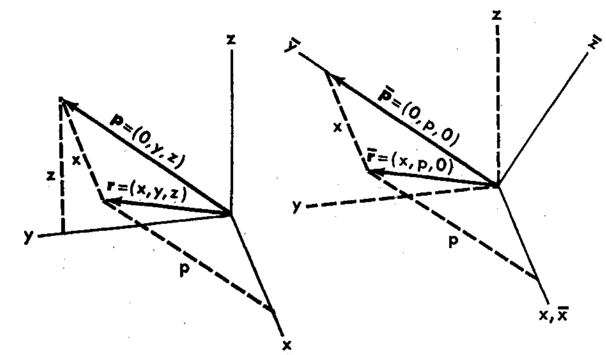

Fig. 3 Rotational coordinate transformation applied to the correlation tensor.

Kronecker delta

$$\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \quad (8)$$

By applying Eqs. (1) and (8) to Eq. (7), it is easily verified that

$$R_{ij}(\mathbf{r}',t) = (-1)^{i+j-2}R_{ij}(\mathbf{r},t) \quad (9)$$

Furthermore, since the reflection operation is distributive,

$$\mathbf{r}'_n - \mathbf{r}'_m = \mathbf{r}'_{mn} \quad (10)$$
$$\mathbf{r}'_n - \mathbf{r}_m = \mathbf{s}'_{mn}$$

where $\mathbf{s}_{mn} = \mathbf{r}_n - \mathbf{r}'_m = (x_n - x_m, y_n + y_m, z_n - z_m)$ denotes the separation vector between the points $\mathbf{r}'_m$ and $\mathbf{r}_n$. These relations are illustrated in Fig. 2. By virtue of Eqs. (1), (9), and (10), the velocity products of like and unlike symmetry of Eq. (4), etc., then reduce to

$$\langle u^{\pm}{}_i(\mathbf{V}t+\mathbf{r}_m,t)u^{\pm}{}_j(\mathbf{V}t+\mathbf{r}_n,t+\tau)\rangle_t = [R_{ij}(\mathbf{r}_{mn},\tau) \pm (-1)^{i-1}R_{ij}(\mathbf{s}_{mn},\tau)]/2 \quad (11)$$
$$\langle u^{\pm}{}_i(\mathbf{V}t+\mathbf{r}_m,t)u^{\mp}{}_j(\mathbf{V}t+\mathbf{r}_n,t+\tau)\rangle_t = 0$$

III. Properties of the Correlation Tensor

Additional properties of the correlation tensor will now be derived for use in evaluating gust velocity cross spectral densities in Sec. IV. Accordingly, by the procedure illustrated in Fig. 3, the correlation tensor $R_{ij} = R_{ij}(\mathbf{r},t)$ may be conveniently expressed in terms of the four distinct components of its representation $\bar{R}_{ij} = R_{ij}(\bar{\mathbf{r}},t)$ relative to the coordinate frame obtained by rotating the original frame about the x axis so that $\bar{z}$ vanishes. Thus,

$$R_{11} = \bar{R}_{11},\ R_{22} = (y/p)^2\bar{R}_{22} + (z/p)^2\bar{R}_{33}$$
$$R_{33} = (z/p)^2\bar{R}_{22} + (y/p)^2\bar{R}_{33},\ R_{21} = R_{12} = (y/p)\bar{R}_{21} \quad (12)$$
$$R_{31} = R_{13} = (z/p)\bar{R}_{21},\ R_{32} = R_{23} = (yz/p^2)(\bar{R}_{22} - \bar{R}_{33})$$

where $\mathbf{r} = (x,y,z)$, $\bar{\mathbf{r}} = (x,p,0)$, and $\mathbf{p} = (0,y,z)$, the vector projection of $\mathbf{r}$ upon the y,z plane.

The general expression for $R_{ij}(\mathbf{r},t)$ given by Eq. (7) may be cast into a more useful form by setting $i = j$ and orienting the velocity component first parallel and then perpendicular to $\mathbf{r}$. These orientations define, respectively, the longitudinal and transverse correlation functions

$$f(r,t) = [r^2F(r,t) + G(r,t)]/u^2$$
$$g(r,t) = G(r,t)/u^2 \quad (13)$$

The functions have been normalized to unity at $r = t = 0$ by dividing through by the mean square gust velocity component, u^2. Equation (7) is then rewritten

$$R_{ij}(\mathbf{r},t) = u^2\{[f(r,t) - g(r,t)]r_ir_j/r^2 + g(r,t)\delta_{ij}\} \quad (14)$$

If we now assume that compressibility effects are negligible in atmospheric turbulence, then the mass continuity condition, represented by the vanishing divergence $\nabla \cdot \mathbf{u}(\mathbf{r},t) = 0$ may be invoked and applied to Eqs. (5) and (14) to yield[4]

$$g(r,t) = f(r,t) + (r/2)\partial f(r,t)/\partial r \quad (15)$$

The conventional turbulence formulations[5] of von Kármán and Dryden assume incompressibility as well as isotropy, and therefore furnish correlation functions that satisfy Eq. (15).

IV. Gust Velocity Cross Spectra

Assume that the corresponding axes of the aircraft and inertial frames coincide at $t = 0$, and that the aircraft subsequently travels in the negative x direction with constant velocity, $\mathbf{V} = (-V,0,0)$. Aerodynamic surface points $\mathbf{r}_m$ and $\mathbf{r}_n$ measured with respect to the aircraft frame, move along the trajectories $\mathbf{r}_m + \mathbf{V}t$ and $\mathbf{r}_n + \mathbf{V}t$ relative to the inertial frame, so that the gust velocities encountered at the surface points are given by $\mathbf{u}(\mathbf{r}_m + \mathbf{V}t,t)$ and $\mathbf{u}(\mathbf{r}_n + \mathbf{V}t,t)$, respectively. Then by definition, the time lag cross correlation function[6] between velocity components i and j measured at the respective surface points may be written

$$R_{ij}(\mathbf{r}_{mn} + \mathbf{V}\tau,\tau) = \langle u_i(\mathbf{r}_m + \mathbf{V}t,t)u_j(\mathbf{r}_n + \mathbf{V}(t + \tau),t + \tau)\rangle_t \quad (16)$$

The corresponding cross spectral density is given by

$$\Phi_{ij}(\mathbf{r}_{mn},f) = 2\int_{-\infty}^{\infty} R_{ij}(\mathbf{r}_{mn} + \mathbf{V}\tau,\tau)e^{-2\pi i f\tau}\, d\tau \quad (17)$$

where f is the frequency in cps.

It can be shown that if V is large enough, then $R_{ij}(\mathbf{r} + \mathbf{V}\tau,\tau)$ is effectively independent of its explicit time argument τ, so that we may set

$$R_{ij}(\mathbf{r} + \mathbf{V}\tau,\tau) = R_{ij}(\mathbf{r} + \mathbf{V}\tau,0) \quad (18)$$

Equation (18) expresses what is commonly termed Taylor's hypothesis,[7] and is valid under typical flight conditions.

Introducing Eq. (18) into Eq. (17) and changing the integration variable to $\tau + x/V$ yields

$$\Phi_{ij}(\mathbf{r},f) = e^{2\pi i f x/V}\Phi_{ij}(\mathbf{p},f) \quad (19)$$

where

$$\Phi_{ij}(\mathbf{p},f) = 2\int_{-\infty}^{\infty} R_{ij}(\mathbf{p} + \mathbf{V}\tau,0)e^{-2\pi i f\tau}\, d\tau \quad (20)$$

and it is recalled that $\mathbf{p} = (0,y,z)$ is the vector projection of $\mathbf{r}$ upon the y,z plane. A normalized cross spectral density in turn may be defined

$$\Psi_{ij}(\mathbf{p},f) = \Phi_{ij}(\mathbf{p},f)/[\Phi_i(f)\Phi_j(f)]^{1/2} \quad (21)$$

where $\Phi_i(f) = \Phi_{ii}(0,f)$ is the power spectral density of the ith gust velocity component, since it represents the cross spectral density between like components at zero separation distance. It then follows from Eqs. (8), (14), (20), and (21), that

$$\Psi_{ij}(0,f) = \delta_{ij} \quad (22)$$

V. Turbulence Description

The frequency domain equivalent of Eqs. (12) may be derived by substituting Eqs. (12), (14), and (20) into (21) to express the normalized gust velocity cross spectra $\Psi_{ij}(\mathbf{p},f)$ in terms of just four describing functions of the form $\psi_{ij} = \Psi_{ij}(\bar{\mathbf{p}},f)$, where $\bar{\mathbf{p}} = (0,p,0)$ denotes the vector projection of $\bar{\mathbf{r}} = (x,p,0)$ upon the y,z plane. Thus,

$$\begin{aligned}\Psi_{11} &= \psi_{11},\ \Psi_{22} = \psi_{33} + (y/p)^2(\psi_{22} - \psi_{33})\\ \Psi_{33} &= \psi_{33} + (z/p)^2(\psi_{22} - \psi_{33}),\ \Psi_{21} = \Psi_{12} = (y/p)\psi_{21}\\ \Psi_{31} &= \Psi_{13} = (z/p)\psi_{21},\ \Psi_{32} = \Psi_{23} = (yz/p^2)(\psi_{22} - \psi_{33})\end{aligned} \quad (23)$$

ψ_{22} and ψ_{33} have been combined into a single function, so as to cause all terms containing y/p or z/p to vanish in the limit as p approaches zero. If the transverse separation distance p and the frequency f are made dimensionless by measuring them in turbulence scale lengths $\eta = p/L$ and radians per scale length $\kappa = 2\pi fL/V$, respectively, then the describing functions become independent of both velocity V and scale length

$$L = \int_0^{\infty} f(r,0)dr$$

The four describing functions are then given by the following expressions, in which the zero time arguments of the correlation functions have been suppressed, and the definition $\rho = (\xi^2 + \eta^2)^{1/2}$, where $\xi = V\tau/L$, has been introduced:

$$\begin{aligned}\psi_{11} &= 2\int_0^{\infty} [(\xi/\rho)^2f(\rho L) + (\eta/\rho)^2g(\rho L)]\cos\kappa\xi d\xi/\pi\phi_1(\kappa)\\ -i\psi_{21} &= 2\int_0^{\infty}(\xi\eta/\rho^2)f[(\rho L) - g(\rho L)]\sin\kappa\xi d\xi/\pi[\phi_1(\kappa)\phi_2(\kappa)]^{1/2}\\ \psi_{33} &= 2\int_0^{\infty} g(\rho L)\cos\kappa\xi d\xi/\pi\phi_2(\kappa)\\ \psi_{22} - \psi_{33} &= 2\int_0^{\infty} (\eta/\rho)^2[f(\rho L) - g(\rho L)]\cos\kappa\xi d\xi/\pi\phi_2(\kappa)\end{aligned} \quad (24)$$

where $\phi_i(\kappa) = u^{-2}\Phi_i(f)df/d\kappa = (V/2\pi u^2L)\Phi_i(f)$ denotes the power spectral density associated with the frequency κ, and normalized to unit mean square gust velocity component.

The standard turbulence formulations of Dryden and von Kármán consist of correlation functions and their associated power spectra.[5] These may be substituted into Eqs. (24) to obtain specific expressions for the four describing functions, which may be evaluated using available integral tables.[8] In the Dryden case,

$$f(\rho L) = e^{-\rho},\ g(\rho L) = e^{-\rho}(1 - \rho/2) \quad (25)$$

and

$$\pi\phi_1(\kappa) = 2/(1 + \kappa^2),\ \pi\phi_2(\kappa) = (1 + 3\kappa^2)/(1 + \kappa^2)^2 \quad (26)$$

so that

$$\begin{aligned}\psi_{11} &= \mu K_1(\mu) - \mu^2K_0(\mu)/2\\ -i\psi_{21} &= \kappa\mu^2K_1(\mu)/2^{1/2}(1 + 3\kappa^2)^{1/2}\\ \psi_{33} &= \mu K_1(\mu) - \mu^2K_0(\mu)/(1 + 3\kappa^2)\\ \psi_{22} - \psi_{33} &= \mu^2K_0(\mu)(1 + \kappa^2)/(1 + 3\kappa^2)\end{aligned} \quad (27)$$

where $\mu = \eta(1 + \kappa^2)^{1/2}$, and K_n denotes the nth order modified Bessel function of the second kind.

In the von Kármán case, it is convenient to introduce a modified scale of turbulence, $\bar{L} = L\Gamma(\frac{1}{3})/\Gamma(\frac{1}{2})\Gamma(\frac{5}{6}) \simeq 1.339L$, and to define the corresponding $\bar{\rho}$, $\bar{\kappa}$, and $\bar{\mu}$. Then,

$$f(\rho L) = A\bar{\rho}^{1/3}K_{1/3}(\bar{\rho}),\ f(\rho L) = A\bar{\rho}^{1/3}K_{1/3}(\bar{\rho}) - (\bar{\rho}/2)K_{2/3}(\bar{\rho}) \quad (28)$$

where $A = 4^{1/3}/\Gamma(\frac{1}{3}) \simeq 0.5925$, and Γ denotes the gamma function. Similarly,

$$\pi\phi_1(\kappa) = 2/(1 + \bar{\kappa}^2)^{5/6},\ \pi\phi_2(\kappa) = (1 + 8\bar{\kappa}^2/3)/(1 + \bar{\kappa}^2)^{11/6} \quad (29)$$

so that

$$\begin{aligned}\psi_{11} &= B[\bar{\mu}^{5/6}K_{5/6}(\bar{\mu}) - \bar{\mu}^{11/6}K_{1/6}(\bar{\mu})/2]\\ -i\psi_{21} &= B\bar{\kappa}\bar{\mu}^{11/6}K_{5/6}(\bar{\mu})/2^{1/2}(1 + 8\bar{\kappa}^2/3)^{1/2}\\ \psi_{33} &= B[\bar{\mu}^{5/6}K_{5/6}(\bar{\mu}) - \bar{\mu}^{11/6}K_{1/6}(\bar{\mu})/(1 + 8\bar{\kappa}^2/3)]\\ \psi_{22} - \psi_{33} &= B\bar{\mu}^{11/6}K_{1/6}(\bar{\mu})(1 + \bar{\kappa}^2)/(1 + 8\bar{\kappa}^2/3)\end{aligned} \quad (30)$$

where $B = 2^{1/6}/\Gamma(\frac{5}{6}) \simeq 0.9944$. Contour plots of the de-

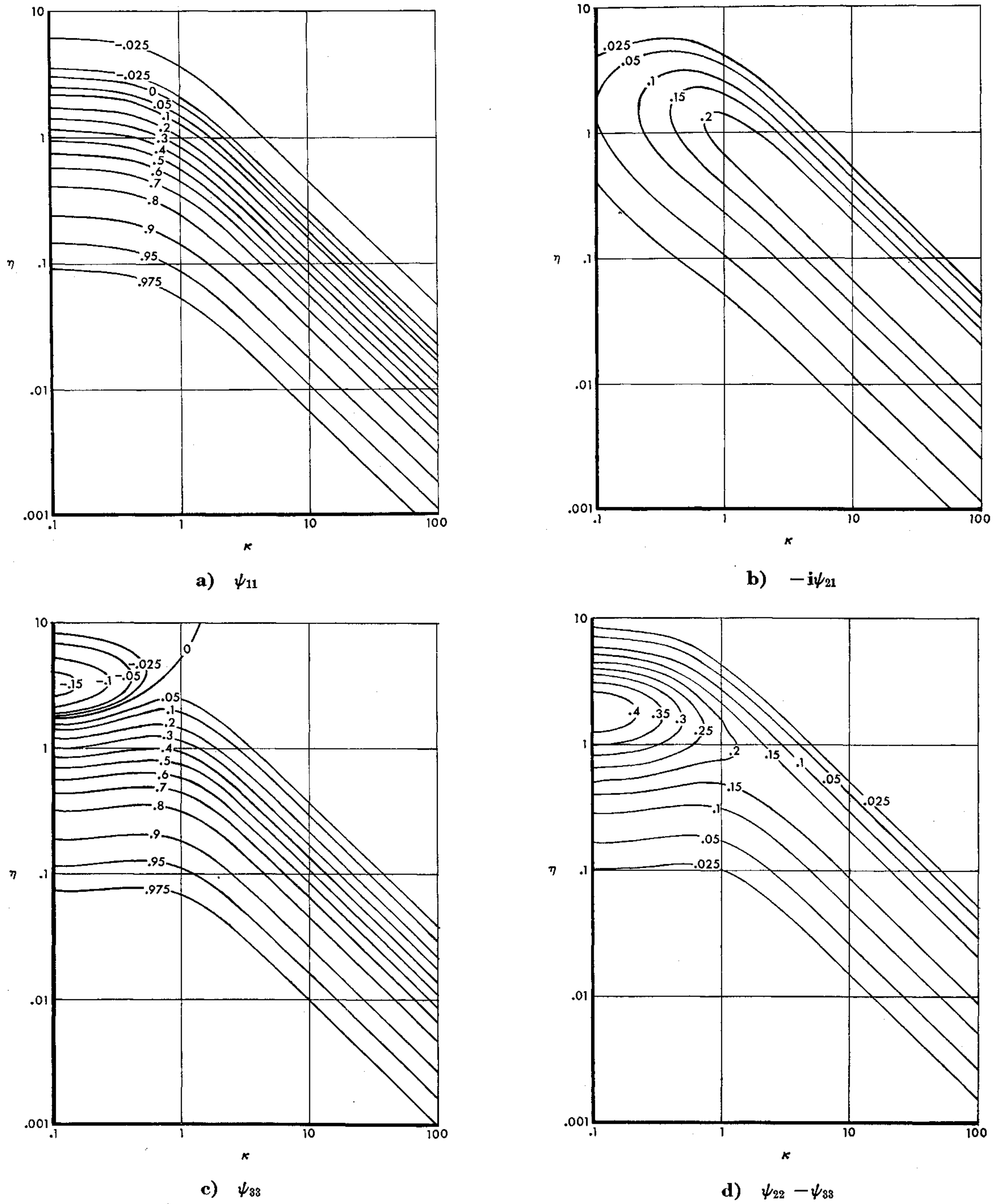

Fig. 4 Contours of the describing functions for the von Karman case plotted vs nondimensional transverse separation $\eta = p/L$ and nondimensional reduced frequency $\kappa = 2\pi fL/V$.

scribing functions for the von Kármán case are shown in Fig. 4. For dynamic response applications, the describing functions given by Eqs. (27) or Eqs. (30) may be stored in compact tabular form for rapid digital computation of the normalized gust velocity cross spectra according to Eqs. (23).

Although the Dryden and von Kármán power spectra differ significantly, contour plots of the corresponding describing functions prove to be almost indistinguishable. Evidently, the describing functions are insensitive to the specific spectral character of the associated turbulence field. This suggests the use of a single representative set of describing functions in all dynamic response calculations, including those in which flight-measured gust input spectra are utilized. Recent atmospheric measurements[9] have tended to yield power spectra which agree with the von Kármán form, indicating that the describing functions given by Eqs. (30) would be most appropriate for this purpose. The usual problem of estimating the scale of turbulence for flight-measured spectra[5] is alleviated inasmuch as dynamic response calculations are typically confined to the asymptotic range ($\eta < 1 < \kappa$) (see Fig. 4), where uncertainty in L merely shifts the point (η,κ) approximately along a contour line, thus, having little effect upon the value of the describing function. This simplification may be utilized even more directly by replacing the describing functions by their appropriate asymptotic forms, which can be derived in the limit as L becomes large and depend only upon p/λ, where $\lambda = V/f$, the gust wavelength.

Unless the transverse dimensions of the aircraft are small compared to the scale of turbulence, three-dimensional effects will prevail even at the lowest frequencies. This conclusion follows from an examination of Eqs. (27) and (30) in the low-frequency limit, and is suggested by reference to the low-frequency range ($\kappa < 1$) in Fig. 4.

VI. Aircraft Response Model

Divide the aerodynamic surfaces into discrete panels which are small enough so that the velocity of the impinging gust is effectively uniform over each, but may be assumed to vary between panels. Let $\mathbf{r}_m = (x_m,y_m,z_m)$ represent the Cartesian components of the aerodynamic center of panel m, and assume that the respective aerodynamic centers define the effective separation vectors between gust components impinging upon different panels. Also let $\mathbf{n}_m = (n_{m1},n_{m2},n_{m3})$ denote the unit vector normal to the aerodynamic reference plane of panel m.

Let $X(t)$ and $X'(t)$ represent corresponding time-dependent dynamic responses occuring within the positive y and negative y half-spaces, respectively, at points which are located by reflection through the x,z plane. Also let $h_m(t)$ and $h'_m(t)$ denote the impulse response functions relating $X(t)$ and $X'(t)$ to $\mathbf{n}_m\cdot\mathbf{u}(\mathbf{r}_m + \mathbf{V}t,t)$, the normal component of the gust velocity impinging upon the mth panel. Finally, if the aircraft configuration is bilaterally symmetric and the primed vector notation is employed, then we may write

$$X(t) = \sum_{m=1}^{N} \int_0^t [\mathbf{n}_m\cdot\mathbf{u}(\mathbf{r}_m + \mathbf{V}(t-\tau),t-\tau)h_m(\tau) + \mathbf{n}'_m\cdot\mathbf{u}(\mathbf{r}'_m + \mathbf{V}(t-\tau),t-\tau)h'_m(\tau)]d\tau \quad (31)$$

$$X'(t) = \sum_{m=1}^{N} \int_0^t [\mathbf{n}_m\cdot\mathbf{u}(\mathbf{r}_m + \mathbf{V}(t-\tau),t-\tau)h'_m(\tau) + \mathbf{n}'_m\cdot\mathbf{u}(\mathbf{r}'_m + \mathbf{V}(t-\tau),t-\tau)h_m(\tau)]d\tau$$

where each side of the aircraft is divided into N panels.

Impulse response functions relating $X(t)$ to the symmetric and antisymmetric two-point gust velocity input configurations defined by Eq. (2) are given by

$$h^{\pm}_m(t) = h_m(t) \pm h'_m(t) \quad (32)$$

so that

$$h_m(t) = [h^+_m(t) + h^-_m(t)]/2$$
$$h'_m(t) = [h^+_m(t) - h^-_m(t)]/2 \quad (33)$$

Let $X^+(t)$ and $X^-(t)$ represent the symmetric and antisymmetric components of $X(t)$. Then,

$$X^{\pm}(t) = [X(t) \pm X'(t)]/2 \quad (34)$$

The reciprocal relations are

$$X(t) = X^+(t) + X^-(t)$$
$$X'(t) = X^+(t) - X^-(t) \quad (35)$$

In anticipation of subsequent analytical requirements, we may extend the lower limits of the time integrals of Eqs. (31) to $-\infty$ on physical grounds that the impulse response function must vanish for future inputs; i.e., $\tau < 0$. The upper limit may in turn be extended to ∞ by stipulating that the input gust velocity vanish for $t < 0$. Application of Eqs. (1, 3, 33, and 35) to Eq. (31) then yields

$$X^{\pm}(t) = \sum_{m=1}^{N} \int_{-\infty}^{\infty} \mathbf{n}_m\cdot\mathbf{u}^{\pm}(\mathbf{r}_m + \mathbf{V}(t-\tau),t-\tau)h^{\pm}_m(\tau)d\tau \quad (36)$$

This convenient dichotomy, whereby symmetric and antisymmetric structural responses are selectively induced by excitations of corresponding symmetry only, prevails throughout the succeeding analytical development and is in accord with conventional structural response formulations.

VII. Response Power Spectrum

The autocorrelation function and power spectral density of the dynamic response $X(t)$ are given by[6]

$$C(\tau) = \langle X(t)X(t+\tau)\rangle_t \quad (37)$$

$$P(f) = 2\int_{-\infty}^{\infty} C(\tau)e^{-2\pi i f\tau}\, d\tau \quad (38)$$

As defined here, $P(f)$, when integrated over half the frequency range, yields the mean square amplitude of $X(t)$; i.e.,

$$\langle X^2(t)\rangle_t = \int_0^{\infty} P(f)df$$

By application of Eqs. (34 and 35) to Eq. (37), $C(\tau)$ becomes

$$C(\tau) = \langle X^+(t)X^+(t+\tau)\rangle_t + \langle X^-(t)X^-(t+\tau)\rangle_t + \tfrac{1}{2}\langle X(t)X(t+\tau)\rangle_t - \tfrac{1}{2}\langle X'(t)X'(t+\tau)\rangle_t \quad (39)$$

The first two terms on the right hand side of Eq. (39) are the autocorrelation functions of the symmetric and antisymmetric components of $X(t)$, and may be denoted by $C^+(\tau)$ and $C^-(\tau)$, respectively, with power spectral densities, $P^+(f)$ and $P^-(f)$. The last two terms cancel through bilateral symmetry, leaving

$$C(\tau) = C^+(\tau) + C^-(\tau) \quad (40)$$

where

$$C^{\pm}(\tau) = \langle X^{\pm}(t)X^{\pm}(t+\tau)\rangle_t \quad (41)$$

and

$$P^{\pm}(f) = 2\int_{-\infty}^{\infty} C^{\pm}(\tau)e^{-2\pi i f\tau}\, d\tau \quad (42)$$

Comparison of Eqs. (38), (40), and (42) shows that

$$P(f) = P^+(f) + P^-(f) \quad (43)$$

Equations (34, 41, and 42) imply that $P^+(f)$ and $P^-(f)$ cannot be isolated unless either the aircraft is symmetrically instrumented to include both $X(t)$ and $X'(t)$, or the response is trivially located in the plane of symmetry, so that $X'(t) = \pm X(t)$.

Substituting Eq. (36) into Eq. (41), expanding the scalar product, interchanging the integrals with the time average, and applying Eqs. (11) then yields

$$C^{\pm}(\tau) = \frac{1}{2}\sum_{n=1}^{N}\sum_{m=1}^{N}\int_{-\infty}^{\infty} d\beta \int_{-\infty}^{\infty} d\alpha h^{\pm}_m(\alpha)h^{\pm}_n(\beta) \times \sum_{j=1}^{3}\sum_{i=1}^{3} n_{mi}n_{nj}[R_{ij}(\mathbf{r}_{mn} + \mathbf{V}(\tau+\alpha-\beta),\tau+\alpha-\beta) \pm (-1)^{i-1}R_{ij}(\mathbf{s}_{mn} + \mathbf{V}(\tau+\alpha-\beta),\tau+\alpha-\beta)] \quad (44)$$

Define frequency response functions,

$$H^{\pm}_n(f) = \int_{-\infty}^{\infty} h^{\pm}_n(t - x_n/V)e^{-2\pi i f t}dt = e^{-2\pi i f x_n/V}\int_{-\infty}^{\infty} h^{\pm}_n(t)e^{-2\pi i f t}\, dt \quad (45)$$

It may be noted that $H^{\pm}_n(f)$ is referenced to the time $t - x_n/V$ at which the y,z plane of the aircraft coordinate system crosses the instantaneous input point, $\mathbf{r}_n + \mathbf{V}t$, as illustrated in Fig. 5. The gust penetration delay due to the longitudinal distance between the input point and the y,z plane is thereby conveniently incorporated into the phase factor appearing before the integral of Eq. (45). Also define normalized symmetric and antisymmetric gust velocity cross spectral densities of the form:

$$\Psi^{\pm}_{ijmn}(f) = [\Psi_{ij}(\mathbf{p}_{mn},f) \pm (-1)^{i-1}\Psi_{ij}(\mathbf{q}_{mn},f)]/2 \quad (46)$$

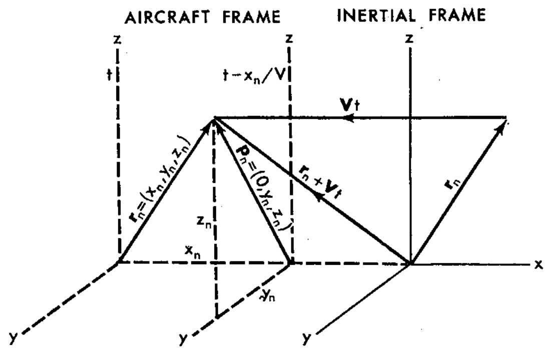

Fig. 5 Referencing of an impulse response function to the time at which the y,z plane of aircraft frame crosses the input point.

where $\mathbf{p}_{mn}$ and $\mathbf{q}_{mn}$ represent the vector projections of $\mathbf{r}_{mn}$ and $\mathbf{s}_{mn}$ upon the y,z plane. Finally, by substituting Eq. (44) into (42), replacing the integration variable τ by $(x_m - x_n)/V + \tau + \alpha - \beta$, and applying Eqs. (17, 19, 21, 45, 46), we obtain

$$P^{\pm}(f) = \sum_{n=1}^{N}\sum_{m=1}^{N} H^{\pm}{}_m{}^*(f)H^{\pm}{}_n(f) \times \sum_{j=1}^{3}\sum_{i=1}^{3} n_{mi}n_{nj}\Psi^{\pm}{}_{ijmn}(f)[\Phi_i(f)\Phi_j(f)]^{1/2} \quad (47)$$

Properties of $\Psi^{\pm}{}_{ijmn}$ relative to gust component interchange and panel interchange will now be established to recast Eq. (47) into a more efficient form for computation. Thus, Eq. (46) implies that

$$\Psi^{\pm}{}_{12mn} = \Psi^{\mp}{}_{21mn},\ \Psi^{\pm}{}_{13mn} = \Psi^{\pm}{}_{31mn},\ \Psi^{\pm}{}_{23mn} = \Psi^{\mp}{}_{32mn} \quad (48)$$

where the frequency argument has been suppressed for brevity. Similarly, Eqs. (23) and (46) together imply that

$$\Psi^{\pm}{}_{iinm} = \Psi^{\pm}{}_{iimn},\ i = 1,2,3 \quad (49)$$

$$\Psi^{\pm}{}_{21nm} = -\Psi^{\mp}{}_{21mn},\ \Psi^{\pm}{}_{31nm} = -\Psi^{\pm}{}_{31mn},\ \Psi^{\pm}{}_{32nm} = \Psi^{\mp}{}_{32mn} \quad (50)$$

$$\Psi^{\pm}{}_{ijmm} = 0 \qquad \text{if } i \neq j \quad (51)$$

Equations (49–50) indicate that interchanging panel subscripts m and n in Eq. (47) has no effect other than to reverse the sign of the imaginary terms (i.e., those containing cross spectral densities between longitudinal and transverse components of the gust velocity). Hence, the range of the sum over m in Eq. (47) may be halved by combining terms involving panel interchange. Using Eq. (51) to identify vanishing terms, and denoting real and imaginary parts of a quantity by $Re(\ )$ and $Im(\ )$, we then obtain:

$$P^{\pm} = \sum_{n=1}^{N} |H^{\pm}{}_n|^2 \sum_{i=1}^{3} n^2{}_{ni}\Psi^{\pm}{}_{iinn}\Phi_i + 2\sum_{n=1}^{N}\sum_{m=1}^{n-1}\Big\{Re(H^{\pm}{}_m{}^*H^{\pm}{}_n)\Big[\sum_{i=1}^{3} n_{mi}n_{ni}\Psi^{\pm}{}_{iimn}\Phi_i + (n_{m2}n_{n3}\Psi^{\mp}{}_{32mn} + n_{m3}n_{n2}\Psi^{\pm}{}_{32mn})(\Phi_3\Phi_2)^{1/2}\Big] + Im(H^{\pm}{}_m{}^*H^{\pm}{}_n)[(n_{m1}n_{n3} + n_{m3}n_{n1})i\Psi^{\pm}{}_{31mn}(\Phi_3\Phi_1)^{1/2} + (n_{m1}n_{n2}i\Psi^{\mp}{}_{21mn} + n_{m2}n_{n1}i\Psi^{\pm}{}_{21mn})(\Phi_2\Phi_1)^{1/2}]\Big\} \quad (52)$$

where

$$Re(H^{\pm}{}_m{}^*H^{\pm}{}_n) = Re(H^{\pm}{}_m)Re(H^{\pm}{}_n) + Im(H^{\pm}{}_m)Im(H^{\pm}{}_n)$$

$$Im(H^{\pm}{}_m{}^*H^{\pm}{}_n) = Re(H^{\pm}{}_m)Re(H^{\pm}{}_n) - Im(H^{\pm}{}_m)Im(H^{\pm}{}_n)$$

In the calculation of the response power spectra by Eq. (52), the subscripts m and n, respectively, range over one-half and one-quarter of the input panels, thereby yielding an eight-fold reduction in computation. Separation of the unit vector components from the frequency response functions results in a further nine-fold reduction. However, since the symmetric and antisymmetric parts of the response must be calculated separately, the effective reduction over-all is 36 fold.

Generalized transfer function matrices of the form $T_{ij}{}^{2\pm}(f) = T_{ji}{}^{2\pm}(f)$ may now be defined which relate the symmetric and antisymmetric parts of $X(t)$ to components i and j of corresponding gust velocity input configurations. Thus,

$$P^{\pm}(f) = \sum_{j=1}^{3}\sum_{i=1}^{3} T_{ij}{}^{2\pm}(f)[\Phi_i(f)\Phi_j(f)]^{1/2} \quad (53)$$

The elements of $T_{ij}{}^{2\pm}(f)$ are readily identified by reference to Eq. (52). It may be observed that the interaction effects between unlike Cartesian components of the gust velocity are embodied in the off-diagonal matrix elements.† By virtue of the asymptotic behavior of the describing functions as discussed in Sec. V, $T_{ij}{}^{2\pm}(f)$ is essentially independent of the scale of turbulence in most flight regimes of interest, and in that case need be computed only once for use with any reasonable gust input spectra.

VIII. Response Cross Spectra

Assume that a gust probe is installed at some point $\mathbf{r}_0 = (x_0,y_0,z_0)$ in the aircraft coordinate system, assigning the panel subscript $m = 0$ to the probe location. Also assume that additional instrumentation is included to provide a time history of the response, $X(t)$, and to furnish aircraft motion data for deriving from probe measurements, a simultaneous three-component gust velocity relative to the inertial system.[10] Then cross correlation functions and cross spectral densities of $X(t)$ with respect to component i of the probe-measured gust velocity may be defined:[6]

$$C_i(\tau) = \langle u_i(\mathbf{r}_0 + \mathbf{V}t,t)X(t + \tau)\rangle_t \quad (54)$$

$$P_i(f) = 2\int_{-\infty}^{\infty} C_i(\tau)e^{-2\pi i f\tau}d\tau \quad (55)$$

in analogy with Eqs. (37) and (38) for the response power spectral case. By the same token, expressions for the cross spectral case can be established without difficulty by methods that essentially duplicate the derivation performed in Sec. VII. Thus, with the aid of Eqs. (2) and (3), (54) can be made to yield the cross spectral analogue of Eq. (40):

$$C_i(\tau) = C^+{}_i(\tau) + C^-{}_i(\tau) \quad (56)$$

where symmetric and antisymmetric cross correlation functions and cross spectra corresponding to Eqs. (41) and (42) are given by

$$C^{\pm}{}_i(\tau) = \langle u^{\pm}{}_i(\mathbf{r}_0 + \mathbf{V}t,t)X^{\pm}(t + \tau)\rangle_t = \langle u_i(\mathbf{r}_0 + \mathbf{V}t,t)X^{\pm}(t + \tau)\rangle_t = \langle u^{\pm}{}_i(\mathbf{r}_0 + \mathbf{V}t,t)X(t + \tau)\rangle_t \quad (57)$$

$$P^{\pm}{}_i(f) = 2\int_{-\infty}^{\infty} C^{\pm}{}_i(\tau)e^{-2\pi i f\tau}\,d\tau \quad (58)$$

† Reference 2 concludes that interaction does not occur by assuming that unlike Cartesian components of the gust velocity are uncorrelated. However, the assumption is generally correct only for components measured at points having no transverse separation, as represented by Eq. (22).

and

$$P_i(f) = P^+{}_i(f) + P^-{}_i(f) \tag{59}$$

in analogy with Eq. (43). Comparison of Eqs. (2), (34), (57), and (58) shows that $P^+{}_i(f)$ and $P^-{}_i(f)$ can be isolated if the aircraft is symmetrically instrumented to furnish both $X(t)$ and $X'(t)$, where a single probe at $\mathbf{r}_0$ is used. Conversely, measurement of $X(t)$ alone is sufficient if a double probe located at $\mathbf{r}_0$ and $\mathbf{r}'_0$ is employed.

Continuing in this fashion, it is easily shown that

$$C^\pm{}_i(\tau) = \frac{1}{2}\sum_{n=1}^{N}\int_{-\infty}^{\infty} d\beta h^\pm{}_n(\beta) \sum_{j=1}^{3} n_{nj}[R_{ij}(\mathbf{r}_{0n} + \mathbf{V}(\tau-\beta),\tau-\beta) \pm (-1)^{i-1}R_{ij}(\mathbf{s}_{0n} + \mathbf{V}(\tau-\beta),\tau-\beta)] \tag{60}$$

which ultimately yields

$$P^\pm{}_i(f) = e^{2\pi i f x_0/V}\sum_{n=1}^{N} H^\pm{}_n(f)\sum_{j=1}^{3} n_{nj}\Psi^\pm{}_{ij0n}(f)[\Phi_i(f)\Phi_j(f)]^{1/2} \tag{61}$$

A two-fold redundancy associated with gust component interchange is present in Eq. (61), but may be accounted for by the use of Eq. (48). Since the panel summation here is single rather than double as in Eq. (52), the computational effort required to obtain cross spectra is trivial compared to that required for the response power spectrum.

Inspection of Eq. (61) shows that symmetric and antisymmetric generalized cross transfer function matrices can be defined, whose elements ij furnish the gain and phase lag with respect to $u^\pm{}_i(\mathbf{r}_0 + \mathbf{V}t,t)$, of that part of $X^\pm(t)$ which results from the collective excitation by component j of all gust inputs:

$$H^\pm{}_{ij}(f) = e^{2\pi i f x_0/V}\sum_{n=1}^{N} H^\pm{}_n(f) n_{nj}\Psi^\pm{}_{ij0n}(f) \tag{62}$$

whereupon,

$$P^\pm{}_i(f) = \sum_{j=1}^{3} H^\pm{}_{ij}(f)[\Phi_i(f)\Phi_j(f)]^{1/2} \tag{63}$$

in analogy with Eq. (53).

Other frequency-dependent functions that are commonly obtained from flight measurements[11] include the cross transfer function, $P_i(f)/\Phi_i(f)$, and the coherence function, $\gamma_i{}^2(f) = |P_i(f)|^2/P(f)\Phi_i(f)$. These provide the gain and phase lag, and the fractional power (mean square amplitude) of that portion of $X(t)$ which is statistically coherent with $u_i(\mathbf{r}_0 + \mathbf{V}t,t)$. Equation (57) implies that the symmetric and antisymmetric parts of the response are correlated only with the corresponding parts of each probe-measured gust velocity component. It is therefore meaningful to define the symmetric and antisymmetric coherence functions:

$$\gamma_i{}^{2\pm}(f) = |P^\pm{}_i(f)|^2/P^\pm(f)\Phi_i(f) \tag{64}$$

The ordinary coherence function can then be expressed as the sum of $\gamma_i{}^{2+}(f)$ and $\gamma_i{}^{2-}(f)$ multiplied, respectively, by the corresponding fraction of $P(f)$ attributable to symmetric and antisymmetric response. Thus,

$$\gamma_i{}^2(f) = \gamma_i{}^{2+}(f)P^+(f)/P(f) + \gamma_i{}^{2-}(f)P^-(f)/P(f) \tag{65}$$

Since unlike Cartesian components of the gust velocity measured at the same point are statistically independent, viz., Eq. (22), those portions of the response that are coherent with the individual probe-measured gust velocity components also are mutually independent. It is therefore appropriate to define a total coherence function as the sum:

$$\gamma^2(f) = \sum_{i=1}^{3}\gamma_i{}^2(f) \tag{66}$$

IX. Probe in Plane of Symmetry

A cap will signify that a quantity has been specialized to the usual case, $y_0 = 0$, in which the gust probe is located in the plane of symmetry of the aircraft. Then $\hat{\mathbf{q}}_{0n} = \hat{\mathbf{p}}_{0n}$, so that Eq. (46) yields:

$$\hat{\Psi}^-{}_{1j0n} = \hat{\Psi}^+{}_{2j0n} = \hat{\Psi}^-{}_{3j0n} = 0,\ j = 1,2,3 \tag{67}$$

Substitution of this result into Eq. (61) reveals that

$$\hat{P}^-{}_1 = \hat{P}^+{}_2 = \hat{P}^-{}_3 = 0 \tag{68}$$

whereupon, Eqs. (59) and (65) reduce to

$$\hat{P}_1 = \hat{P}^+{}_1,\ \hat{P}_2 = \hat{P}^-{}_2,\ \hat{P}_3 = \hat{P}^+{}_3 \tag{69}$$

and

$$\hat{\gamma}_1{}^2 = \hat{\gamma}_1{}^{2+},\ \hat{\gamma}_2{}^2 = \hat{\gamma}_2{}^{2-},\ \hat{\gamma}_3{}^2 = \hat{\gamma}_3{}^{2+} \tag{70}$$

Centering the probe has halved the computation of each cross spectral density $\hat{P}_i$ by restricting it either to symmetric or to antisymmetric response, according to Eqs. (69). Each gust component so measured is coherent only with that portion of the response specified by Eqs. (70). Similarly, application of Eqs. (67) to Eq. (62) reveals that the matrix elements of $\hat{H}^+{}_{ij}$ and $\hat{H}^-{}_{ij}$ vanish in complementary patterns; i.e.,

$$\hat{H}^-{}_{1j} = \hat{H}^+{}_{2j} = \hat{H}^-{}_{3j} = 0,\ j = 1,2,3 \tag{71}$$

X. Small Aircraft Limit

We will now consider response to three-dimensional turbulence in the limit as the transverse dimensions of the aircraft become small. The small aircraft limit corresponds to $p = 0$. If the limit, signified by inserting a tilde, is applied to Eqs. (24), then (23) and (24) reduce to $\tilde{\Psi}_{ij} = \tilde{\psi}_{ij} = \delta_{ij}$, so that all of the gust velocity cross spectral densities defined by Eq. (46) vanish except for

$$\tilde{\Psi}^+{}_{11mn} = \tilde{\Psi}^-{}_{22mn} = \tilde{\Psi}^+{}_{33mn} = 1 \tag{72}$$

Upon application of this result to Eqs. (52), (53), and (62–66), the response power spectrum, coherence functions and cross transfer functions reduce to

$$\tilde{P} = \tilde{T}^{2+}{}_{11}\Phi_1 + \tilde{T}^{2-}{}_{22}\Phi_2 + \tilde{T}^{2+}{}_{33}\Phi_3 \tag{73}$$

$$\tilde{\gamma}_1{}^2 = \tilde{T}^{2+}{}_{11}\Phi_1/\tilde{P},\ \tilde{\gamma}_2{}^2 = \tilde{T}^{2-}{}_{222}/\Phi\tilde{P},\ \tilde{\gamma}_3{}^2 = \tilde{T}^{2+}{}_{33}\Phi_3/\tilde{P} \tag{74}$$

$$\tilde{\gamma}^2 = 1 \tag{75}$$

$$\tilde{P}_1/\Phi_1 = \tilde{H}^+{}_{11},\ \tilde{P}_2/\Phi_2 = \tilde{H}^-{}_{22},\ \tilde{P}_3/\Phi_3 = \tilde{H}^+{}_{33} \tag{76}$$

where

$$\tilde{T}^{2+}{}_{11} = |\tilde{H}^+{}_{11}|^2,\ \tilde{T}^{2-}{}_{22} = |\tilde{H}^-{}_{22}|^2,\ \tilde{T}^{2+}{}_{33} = |\tilde{H}^+{}_{33}|^2 \tag{77}$$

and

$$\begin{bmatrix}\tilde{H}^+{}_{11}\\ \tilde{H}^-{}_{22}\\ \tilde{H}^+{}_{33}\end{bmatrix} = e^{2\pi i f x_0/V}\begin{bmatrix}\sum_{n=1}^{N} H^+{}_n n_{n1}\\ \sum_{n=1}^{N} H^-{}_n n_{n2}\\ \sum_{n=1}^{N} H^+{}_n n_{n3}\end{bmatrix} \tag{78}$$

All matrix elements not included in Eqs. (77) have vanished, showing that responses associated with each gust component are either symmetric or antisymmetric, and that interaction effects between unlike gust components have disappeared. Furthermore, the generalized transfer functions and the moduli of the corresponding generalized cross transfer functions are now identical. Substitution of Eqs. (78) into (77) reveals that $T_{ij}{}^{2\pm}$ has degenerated from a quadratic form involving a double sum over panels, to the modulus squared of a single sum.

Significantly, Eqs. (76–78), which were derived by applying the small aircraft limit to the three-dimensional gust response formulation, are identical to the expressions obtained under the familiar one-dimensional turbulence approximation in which the transverse spatial dependence of the turbulence velocity field is ignored, and only spatial dependence along the flight path is retained. Moreover, Eqs. (73–75) establish an analytical justification, heretofore unavailable, for superimposing the response power spectra arising from the three gust components and ignoring interaction effects under the conventional turbulence approximation.

XI. Summary and Discussion

An improved method for analyzing the dynamic response of an aircraft to three-dimensional isotropic turbulence has been presented. The new formulation utilizes four describing functions to determine normalized gust velocity cross spectra for the turbulence field. The describing functions, which depend upon gust frequency and interpanel separation, are expressed in compact dimensionless form for rapid evaluation of the gust cross spectra. If the fluctuating longitudinal component of the gust velocity is omitted, as is common, then only two of the four describing functions (i.e., ψ_{33} and $\psi_{22} - \psi_{33}$) are required, and the 3×3 matrix expressions appearing throughout the formulation are correspondingly reduced to 2×2 size. Describing functions are developed for both the Dryden and von Kármán spectra and may be used with gust input consisting of either analytical or measured spectra.

It is shown that in the limit as the transverse dimensions of the aircraft become small, the three-dimensional formulation does indeed reduce to the current one-dimensional analytical model: 1) interactions disappear between unlike Cartesian components of the gust velocity; 2) longitudinal and vertical gust inputs excite only symmetric responses; 3) lateral gust inputs induce only antisymmetric responses; 4) like components of the gust velocity become fully coherent, thereby reducing the quadratic form for each response power spectrum to the squared moduli of a simple sums.

Conventional dynamic response computer programs may be converted to the new formulation and applied to the dynamic response test of an aircraft. Thus, frequency-dependent functions obtained by spectral analysis of data recorded during test flights in turbulence may be compared with corresponding analytical results, utilizing as input the gust spectra measured during the test. Such comparisons have been obtained by the author and will be presented in a following paper.

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